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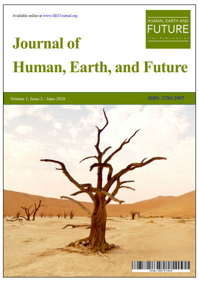

General Features of the q-XY Opinion Model

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Abstract

In this article, we will discuss the general properties of the q-opinion model, which is based on the extension of the XY magnet model. After considering a short briefing of our recently introduced q-XY model and providing the general statistical mechanics calculation for it, we analysed the specifics and application of the 2-node chain. The model is capable of interpreting some particular features of the opinions of a couple, including shifting behavior, losing interest in a given issue for specific opinion microstates, etc. We proposed to use this model in a modified version of the preferential attachment rule for link establishment in a social network.

Keywords: Opinion Dynamics; XY Magnets; q-XY Opinion; Statistical Mechanics.

1. Introduction

The use of magnetic models in the study of opinion systems has demonstrated the effectiveness of the use of physical models to explain society and specific behavioural features qualitatively and quantitatively. In some applications, as in voter models [1, 2], and Potts variants [3], opinion entities are represented by Ising's spin-like variables which live in a network. In other cases, it is proposed the use of an energy-like function as a utility to describe a motivation quantity as described by Stauffer (2012) [4]. Advanced consideration physical of socio-physics background have been presented in the framework of network sciences [5-7] or under complexity context, [8-10], etc. Herein we will introduce a XY classical magnet-like opinion model to describe specific features of the agreement in very small groups of individuals, say couples, family members, friends, opponents, organization and community leaders, etc., or even duos of discussants who share common interests or may act upon a clearly defined issue. Specifically, we will consider some specific features of opinion formation that are common in reality but not considered in particular by standard opinion formation literature. So, during a conversation, it is likely that an agreement would be reached following pathways given in continuous models [11–15] as many others, etc., but also the opinion could shift abruptly, or more generally, the agreement would not follow quantitatively a smooth and monotone footpath. A briefing of the idea that motivated our model has been provided in the next paragraph.

2. Brief Review of the Ultra-small Cluster of 2D XY Magnet Dimer Model

The XY magnet is based in the classical magnet models developed by W. Heisenberg. Within the general framework of the classical XY spin model, the spins occupy the sites of a regular lattice in an arbitrary dimension [16]. In the 3D XY model, each spin vector is free to point in directions. (θ, φ) . In the 2D model each XY spin, $\vec{S}_i$ is

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represented as a 2D unit-length vector, $\vec{S} = (S_x, S_y)$ and is free to point along any direction on the 2D plane. Recently, the studies for XY magnet properties have known significant increase due to the direct application for the study of ferromagnetism in the layers of some compounds as K_2CuF_4, Rb_2CrC_{l4}, or $CoCl_2$ intercalated in graphite for structures as Gd_2CuO_4 [$YBa_2Cu_3O_6$+x], layered compound $BaCo_2(AsO_4)$, the intermolecular magnetic some specific dimers molecule [16] and many others. In the modelling for those applications, the individual magnetic spins were taken units vector by rescaling them by the factor $\hbar\sqrt{S(S+1)}$, By limiting the interaction to the first neighbours in the lattice, the coupling constant J also is rescaled to ± 1, where the signs + is taken for ferromagnetic and (-) for antiferromagnetic. Similarly, to the other spin models, boundary condition may apply and the calculation follow the standard statistical physics pathways: firstly, one calculates the partition's function $Z = \int exp(-\beta H)\, d\Gamma$ and next the thermodynamic quantities can be evaluated straightforwardly by employing the thermodynamic statistics equations. So, for the ultra-small dimer XY magnet of two entities, the partition function has the form;

$$Z = \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2\, exp[-\beta(-J\,cos(\varphi_2 - \varphi_1) - Bcos\varphi_1 - Bcos\varphi_2] \tag{1}$$

And for a N spins chain the partition function is;

$$Z = \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \ldots \int_0^{2\pi} d\varphi_n\, exp[-\beta \sum_{i=1}^{N}(-J\,cos(\varphi_{i+1} - \varphi_i) - Bcos\varphi_{i+1} - Bcos\varphi_i)] \tag{2}$$

Ciftja et al. (1999) [17] proposed an analytical calculating pathway by introducing $\vec{S} = \sum_{i=1}^{N} \vec{S}_i$ to transform the Hamiltonian onto $H = -\frac{J}{2}(S^2 - N) - \mu\vec{B}\vec{S}$. Next by using the N-delta Dirac function proposed in [17, 18] the thermodynamical quantities have been evaluated for various XY structures. Specifically, a full development of this calculation based on Equation 2 is presented in Ciftja & Prenga (2016) [16].

3. The Extension of the XY Classical Magnets to the Opinion Modelling

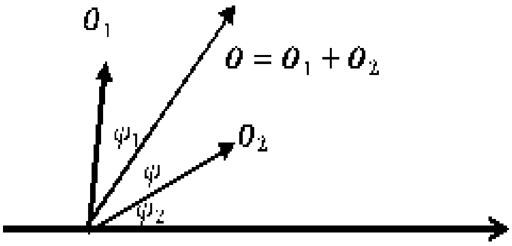
Let consider the duos of opinion entities that entertained an agreement process. They are influenced by exterior disturbances T, and suppose that the pair is asked about an issue embodied in the exterior field F. Assume also that the pairs needs to act on-block, hence a strong interest exits between each one. By a straightforward analogy, we proposed to approach this system by starting from the classical XY-magnet model [19]. We start from the largely accepted metrics on the opinions models that assign opinion values in the segment [-1,1], similarly as Deffuant et al. [12] etc., say from fully dissimilarity to fully accordance. The opinion entities are represented therefore by the vector O_1, O_2 of the unity magnitude, so the agreement in the pair is given by the vector $(\vec{O}_1 + \vec{O}_2)/2$. It ranges from 0 to 1. Individual accord for an exterior issues F is its projection $Ocos\varphi = cos\varphi$. This vector model is able to capture the nuances of the behaviour for an opinion pair: the individual supports for the issues F are usually different, but however, the solid pair act as unique opinion and attribute a given unique support, after an agreement process has been accomplished. Clearly the vector sum fulfils this requirement. Note that out of the pair, the two individuals could present their own opinion which differs from the one represented by the sum of the vectors. More arguments for this choice are presented in our recent works [19, 20]. Next, we assumed that the agreement is practically utilitarian in the sense that peoples acts rationally, if something worth, produced a benefit or satisfaction, it is embraced, otherwise it would be rejected. So the model should incorporates the energy of the system according to Stauffer (2012) [4], say the interests, the satisfaction, or the unhappiness as driven motor for an opinion process. The Hamiltonian of the system is used to describe the dissatisfaction, the temperature encompasses all disturbing effects except opinion ones and the exterior field represent the intensity of the issue under discussion, in accordance with idea presented by Stauffer (2012) [4], and likewise other similar consideration [19-21] or general agent-based opinion modelling. In our model, the x-axis is taken along the exterior field F whereas y-direction represents the "*no-interests in F*" state [20, 22]. Alike in statistical physics we assume that optimization utility acts as driving mechanisms for the dynamics of the opinion system. Initially we guess that the utility (unhappiness) would contain the classical Hamiltonian term say the matching interest between nodes $h_{matching} \sim \vec{O}_1\vec{O}_2$ and the fitting with exterior issue $h_{fitting} \sim \vec{F}\,\vec{O}_1 + \vec{F}\,\vec{O}_1$, so the reference utility;

$$U = -J\vec{O}_1\vec{O}_2 - \vec{F}\,\vec{O}_1 - \vec{F}\,\vec{O}_2 \tag{3}$$

Now let makes the model more realistic. Consider the shifting attitude during the agreement which could not be explain by the XY Hamiltonian, because there is no such behaviour for average magnetism $m_{average} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial B}$ which is physical analogue of the average opinion (here Z is the partition function and β is the thermodynamic temperature). Also, it can happen that the interest encompassed on the strict pair's compliance, can be paired with the interest toward the outer issue F in a specific way that the result could change substantially. It can happen also that during a tense debate, the product would give no agreement toward the issues F even the couple has a good inner-agreement, that is, both peoples agree to not agree at all with the question F. Those specific behaviours cannot be reproduced by a rude adaption of the XY model. The orthodox adaption of physical models results inappropriate, but also it has good premises to explain other opinion outcomes. Bringing all together, in our previous works [5, 6, 20] we have proposed the opinion utility function in the form;

$$U = -J\sum_{i\neq j} \vec{O}_i\vec{O}_j - \vec{F}\sum \vec{O}_i + q\left(-J\sum_{i\neq j} \vec{O}_i\vec{O}_j\right)\left(-\vec{F}\sum \vec{O}_i\right) \tag{4}$$

Herein, q is the pairing intensity between node-interaction utility and the utility related to the interaction of total opinion with the exterior field. Apparently, the extra utility artifice acts as destabilising term and modifies the total utility substantially. Particularly, it demonstrates the capability to explain the dependency of the agreement dynamics from the interior pre-conditions, the shifting of the agreement behaviour, the existence of specific regimes of the behaviour etc. This can be easily understood by analysing the system in the zero-temperature limit, hence, by exploring the behaviour of Equation 1. In Dorogovtsev et al. (2000) [6] study we have described the process as "*the early stage of opinion formation*" and in Albert & Barabási (2002) [5] study, we used the opinion calculated by this model as the starting configuration when using Deffuant update mechanism described in Albi et al. (2016) [9]. Flache & Hegselmann [15] used the name *q-XY opinion* for this model. Note that other process could happen as by other contacts of the peoples, so the state could not be final, and therefore we called this "early stage of opinion formation". So far, starting form a random configuration, by assuming thermodynamic time the pair reach an agreement $\vec{O}_{avg} = \langle(\vec{O}_1 + \vec{O}_2)/2\rangle_{T,F,q}$ and, a support to the question F given by $((\vec{O}_1 + \vec{O}_2)/2\,)_x = \langle O_x\rangle$. Accordingly, the resulting vector opinion produces various scenarios.



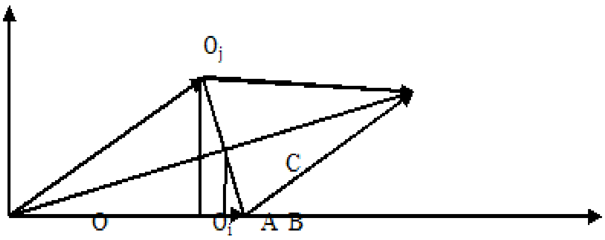


Figure 1. Opinion vectors. Left frame, general view. Right frame, total opinion is rotated so the militant opinion is aligned parallel to the exterior field.

In the $q = 0$ limit we have XY classical magnet which still comprehends intriguing behavior.

3.1. The 1D Chain q-XY Model

In principle, the opinion values in (2) can be calculated by using statistical physics methodology. For easiness of the reading we are presenting the calculation steps, which despite present notation are like the calculation in [16] or [17] , with some modification due to the utility form (1). Let start from the XY magnet-Hamilton utility $U_{XY} = -J\sum_{i\neq j}^{2}\vec{O}_i\vec{O}_j - \vec{F}\sum\vec{O}_i$ and replace $\vec{O}_1\vec{O}_2 = \frac{(\vec{O}_1+\vec{O}_2)^2-\vec{O}_1^2-\vec{O}_1^2}{2} = \frac{\vec{O}^2-2}{2}$.

Considering $\vec{O}_1^2 = \vec{O}_1^2 = 1$ the utility function $U_{XY} = -J\frac{\vec{O}^2-2}{2} - \vec{F}\,\vec{O}$. The variables are $O = \sqrt{(\vec{O}_1 + \vec{O}_2)^2}$, and $\varphi = \frac{arcos(\vec{F}*\vec{O})}{FO}$ and according to [16] or [17] this transformation rends the calculation realizable analytically. Utility (2) takes the form;

$$U = -J\frac{O^2-2}{2} - \vec{F}\vec{O} + qJ\frac{O^2-2}{2}\vec{F}\vec{O} = -J\frac{O^2-2}{2} - \vec{F}\vec{O}(1 - qJ\frac{O^2-2}{2}) \tag{5}$$

The partition function for the 2-node chain reads;

$$Z = \int_{\Gamma_O} e^{\beta\left(\frac{J(O^2-2)}{2}+\vec{F}\vec{O}\left(1-q\left(\frac{J(O^2-2)}{2}\right)\right)\right)} d\Gamma_O \tag{6}$$

Notice N-dimension Dirac delta function for 2D space [17];

$$\delta^{(N)}(\vec{O} - \vec{O}_1 - \vec{O}_2 - ..\vec{O}_N) = \int exp\left(i\vec{k}(\vec{O} - \vec{O}_1 - \vec{S}_2 - ..\vec{O}_N)\right)\frac{d^2\vec{k}}{(2\pi)2}$$

For N=2 we have;

$$1 = \int_{(O^2)} d^2O\,\delta^2(\vec{O} - \textstyle\sum_{j=1}^{2}\vec{O_j}) = \int_{O^2} d^2O\int_0^\infty \frac{d^2k}{(2\pi)^2}e^{i\vec{k}(\vec{O}-\vec{O_1}-\vec{O_2})} \tag{7}$$

Putting this expression in (6) and noticing that $d\Gamma_O = d^2O d\varphi_1 d\varphi_2\, d^2k$, the partition functions is:

$$Z = (2\pi)^2\int_0^2 OdO\int_0^{2\pi} d\varphi_O\int_0^{2\pi} d\varphi_1\int_0^{2\pi} d\varphi_2\int_0^\infty \frac{d^2k}{(2\pi)^2}e^{i\vec{k}\left(\vec{O}-\sum_{j=1}^{2}\vec{O_j}\right)}e^{\beta\left(\frac{J(O^2-2)}{2}+\vec{F}\vec{O}\left(1-q\left(\frac{J(O^2-2)}{2}\right)\right)\right)} \tag{8}$$

Note that in the O space we have $d^2O = OdOd\theta_o$. Next, assign $\vec{k}\vec{O}_i = kcos\varphi_i$, and use:

$$\int_0^{2\pi} d\phi_j * e^{\pm ix\cos\phi_{ji}} = 2\pi J_0(x); x \geq ,0, \quad (9)$$

The magnitude of opinion vector is $O_1 = O_2 = 1$, therefore we gat:

$$\int_0^{2\pi} e^{ikcos(\varphi_1)}\ d\varphi_1 = \int_0^{2\pi} e^{ikcos(\varphi_2)}\ d\varphi_2 = 2\pi J_0(k) \quad (10)$$

$$\int_0^{2\pi} e^{ikOcos(\varphi_0)}\ d\varphi_0 = 2\pi J_0(Ok) \quad (11)$$

where $J_n(x)$ are the first kind Bessel functions of the order n. In Hegselmann & Krause [13] it has been found that:

$$\int_0^{\infty} kdk\, J_0(k)^2 J_0(Ok) = \frac{2}{\pi}\frac{1}{O\sqrt{4-O^2}}\ for\, 0 \leq O \leq 2 \quad (12)$$

Next, we calculate;

$$\int_0^{2\pi} d\varphi_O\, e^{FO\beta\left(1-q\frac{J(O^2-2)}{2}\right)cos\varphi_O} = 2\pi I_0\left(\beta FO\left(1-\frac{qJ}{2}(O^2-2)\right)\right) \quad (13)$$

where $I_n(x)$ are the incomplete Bessel function of order n. The Z-function (3) has the analytic form:

$$Z = 8\pi e^{-\beta}\int_0^2 e^{\frac{J\beta O^2}{2}} I_0\left(\beta FO\left(1-\frac{qJ}{2}(O^2-2)\right)\right)\frac{1}{\sqrt{4-O^2}}dO \quad (14)$$

Now, averaged quantities are obtained by using thermodynamics equations similarly with common magnetic systems. The average opinion induced in the system on a given state $\{J,\beta,F,q\}$ is;

$$\langle O\rangle_F = \frac{1}{\beta}\frac{\partial ln(Z)}{\partial F} = \frac{\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{1}{\sqrt{4-O^2}} I_1\left(\beta FO\left(1-\frac{qJ}{2}(O^2-2)\right)\right) * O*\left(1-\frac{qJ}{2}(O^2-2)\right)dO}{\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{1}{\sqrt{4-O^2}} I_0(\beta FO\left(1-\frac{qJ}{2}(O^2-2)\ \right)dO} \quad (15)$$

The average utility per opinion node;

$$U = \frac{\partial lnZ}{\partial\beta} = J - \frac{\frac{J}{2}\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{1}{\sqrt{4-O^2}} I_1(\beta FA(O)) * O^2 dO - F\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{1}{\sqrt{4-O^2}} I_1(\beta FA(O)) * A(O)dO}{\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{1}{\sqrt{4-O^2}} I_0(\beta F\, A(O)dO} \quad (16)$$

where we used $A(O) = O\left(1-\frac{qJ}{2}(O^2-2)\right)$. The analytic results (12), (13) etc., are important because they permit us to perform numerical calculation for quantitative evidences. Another interesting parameter arising from such analysis could be the correlation function, but in the 2-node chain it is not so important. The correlation function is

$$\langle O_1 O_2\rangle = \frac{1}{\beta}\frac{\partial}{\partial J} ln\, Z_{2,\alpha} = \frac{\int_0^2 e^{\frac{J\beta O^2}{2}}\left[\frac{O^2}{2}\frac{I_0(\beta FA(O))}{\sqrt{4-O^2}} + \frac{I_1(\beta FA(O))}{\sqrt{4-O^2}} F\left(-\frac{\alpha(O^2-2)}{2}\right)\right]dO}{\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{I_0(\beta FA(O))}{\sqrt{4-O^2}}dO} \quad (17)$$

The susceptibility is important. It is given by $\chi(\beta, F=0) = \lim_{F\to 0}\frac{\partial}{\partial F}\langle O_x\rangle$, so we can calculate it by performing the derivative of (12), providing that no special points exist. However, for the zero-field limit we observe that q-opinion become a XY magnet object and in this case, in similarity with calculation provided in the reference for classical magnets in zero field [16] we have;

$$\chi_{F=0} = 1 + \langle O_1 O_2\rangle_{F=0} = \frac{\int_0^2 e^{\frac{J\beta O^2}{2}}\left[\frac{O^2}{2}\frac{I_0(\beta FA)}{\sqrt{4-O^2}} + \frac{I_1(\beta FA)}{\sqrt{4-O^2}} F\left(-\frac{\alpha(O^2-2)}{2}\right)\right]dO}{\int_0^2 e^{\frac{J\beta O^2}{2}}\frac{I_0(\beta FB)}{\sqrt{4-O^2}}dO} \quad (18)$$

3.2. Three Nodes q-opinion

Let consider the bounded chain. The Hamiltonian is;

$$H_3(\vec{F}, J) = -\frac{J(O^2-3)}{2} - \vec{F}\vec{O} + q * \frac{J(O^2-3)}{2} * \vec{F}\vec{O} \quad (19)$$

So

$$Z_3(F,J,T) = \int d\Gamma\, exp\left(-\beta * \left[\frac{J(O^2-3)}{2} - \vec{F}O + q * \frac{J(O^2-3)}{2} * \vec{F}\vec{O}\right]\right)$$

Following the same trick proposed for the tow node system, we get;

$$Z_3 = \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{2\pi} d\phi_3 \int_0^{3} d^2O \frac{d^2k}{(2\pi)^2} e^{i\vec{k}(\vec{O}-\vec{O}_1-\vec{O}_2-O_3)} e^{-\beta\left[\frac{J(O^2-3)}{2}-\vec{F}O+q*\frac{J(O^2-3)}{2}*\vec{F}O\right]}$$

$$= \int_0^{\infty} \frac{d^2k}{(2\pi)^2} \int_0^{2\pi} d\phi_1\, e^{i\vec{k}\vec{O}_1} \int_0^{2\pi} d\phi_2 e^{i\vec{k}\vec{O}_2} \int_0^{2\pi} d\phi_3 e^{i\vec{k}\vec{O}_3} \int_0^{2\pi} e^{i\vec{k}\vec{O}}\, d\varphi_O e^{-\left[\frac{J(O^2-3)}{2}-\vec{F}\vec{O}+q*\frac{J(O^2-3)}{2}*\vec{F}\vec{O}\right]} OdO$$

Now, considering again that $O_1 = O_2 = O_3 = 1$, we have;

$$\int_0^{2\pi} d\phi_1\, e^{i\vec{k}\vec{O}_1} = \int_0^{2\pi} d\phi_2 e^{i\vec{k}\vec{O}_2} = \int_0^{2\pi} d\phi_3 e^{i\vec{k}\vec{O}_3} = \int_0^{2\pi} d\phi e^{ikcos\varphi} \equiv 2\pi J_o(k)$$

$$\int_0^{2\pi} e^{i\vec{k}\vec{O}}\, d\varphi_O = \int_0^{2\pi} e^{ikOcos(\varphi_O)}\, d\varphi_{O=}2\pi J_o(O*k) \tag{20}$$

Next, we write;

$$\int_0^{3} e^{-\beta\frac{J(O^2-3)}{2}} OdO \int_0^{2\pi} e^{-\beta FO\left[1-q*\frac{J(O^2-3)}{2}\right]cos(\theta)}\, d\theta = \frac{d^2k}{(2\pi)^2} \int_0^{3} e^{-\beta\frac{J(O^2-3)}{2}} I_0\left(\beta FO\left[1-q*\frac{J(O^2-3)}{2}\right]\right) OdO$$

So the final form for partitions function is;

$$Z = 2(2\pi)^2 \int_0^{\infty} kdk(J_0(k))^3 J_0(kO) \int_0^{3} e^{-\beta\frac{J(O^2-3)}{2}} I_0\left(\beta FO\left[1-q*\frac{J(O^2-3)}{2}\right]\right) OdO \tag{21}$$

A plausible form of the partition form could be reached by trying to integrate each part but with our best effort we did not obtained any analytic form for the integral $\int_0^{\infty} k\mathrm{dk}\left(\mathrm{J}_0(\mathrm{k})\right)^3 \mathrm{J}_0(\mathrm{kO})$. However, for numerical calculation we remark the convergent nature. Regarding the last integral in we can make use of the infinite series form of the Bessel function of first kind. By using $J_\nu(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(\nu+l+1)} \left(\frac{x}{2}\right)^{2l+\nu}$ and noting that $\Gamma(\nu+l+1) = (\nu+l)!$, we have evaluated that;

$$\int kdkJ_0(kO)[J_0(k)]^3 = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \int kdk\left[(-1)^l \frac{k^{2l}O^{2l}}{4^l(l!)^{2l}}\right]\left[(-1)^m \frac{k^{2m}}{4^m(m!)^{2m}}\right]^3$$

Has no closed form solution. However, due to its convergent nature, it can be solved numerically. The other form also can be estimated as non-diverging term so in principle we can perfmon numerical calculation. The averaged opinion induced by exterior field would be;

$$\langle O \rangle = \frac{\int_0^{\infty} k(J_0(k))^3 J_0(kO)dk \int_0^3 e^{-\beta\frac{J(O^2-3)}{2}} I_1\beta\left(FO\left[1-q*\frac{J(O^2-3)}{2}\right]\right) FO\left[1-q*\frac{J(O^2-3)}{2}\right] OdO}{\int_0^{\infty} k(J_0(k))^3 J_0(kO)dk \int_0^3 e^{-\beta\frac{J(O^2-3)}{2}} I_0\left(\beta FO\left[1-q*\frac{J(O^2-3)}{2}\right]\right) OdO} \tag{22}$$

We considered the calculation with this last form and the fuzzy result confirmed the idea that in e general temperature, the triples of q-opinion seem to be unstable system. However, the limit forms of low temperature, strong field etc., could very interesting and we will address in the future.

4. Behaviour of the q-opinion

After assessing analytic form, we can perfmon calculation and quantitative analysis of the model. The commonly physical specific states are the zero-temperature limit, weak and strong field limit, etc. for the zero temperature, we should address in the exact form of the utility.

4.1. The Coupling-type Persevering Conditions

The utility function can be written;

$$U = -\frac{J(1+2q*\vec{F}\vec{O})}{2}\left(\vec{O}^2-2\right) - \vec{F}\vec{O} \equiv \frac{J_{xy}}{2}\left(\vec{O}^2-2\right) - \vec{F}\vec{O} \tag{23}$$

In the Equation 9, we have formally the XY–magnet Hamiltonian where the interaction parameter;

$$J_{new} = \frac{J(1+2q*\vec{F}\vec{O})}{2} \tag{24}$$

Potentially the FM type interaction would become AFM type when the sign of the effective J change. If a steady state is reached, the pair will preserve its initial type if $1 + 2q * \mathrm{FO_x} > 0$.

Typically for the zero-temperature limit, $O_x = J$, so in this case $2q * FJ > 1$. It resulted that the product q*F practically manages the shifting behavior of the couple. For the non-zero temperature we should calculate the averaged opinion and the coupling-type preservation condition reads $q > -\frac{1}{F<O>_x}$ or $F > -\frac{1}{q<O>_x}$ or also $< O >_x > -\frac{1}{qF}$.

Those transcendent conditions can be used in the numerical calculation. It worth to underline that q-opinion model predicts two regimes according to the coupling type, and this is a dynamical property rather static. Depending on the agreement level and the exterior field, a given level on the q-feature of the system will impose the shift on the type pf the coupling. Starting from a very collaborative and the same interest-sharing duos, suddenly, the member of the pair feels themselves as opponent duos and vice versa. If we are convicted that peoples on the couples would resist the shifting tendency imposed by the nature of the system, we should perform calculation by using the transcendent equations mentioned above. Also, for this precondition, there are threshold values of the interest coupling parameter q and exterior field given by relation $qF < O >_x > -1$.

4.2. Interaction Type Preservation

Rewrite the utility in the form;

$$-\frac{J}{2}\left(\vec{O}^2-2\right)-\vec{F}\vec{O}\left(1-\frac{Jq}{2}\left(O^2-2\right)\right) \equiv -\frac{J}{2}\left(\vec{O}^2-2\right)-\mu\vec{F}\vec{O} \tag{25}$$

If $1-\frac{Jq}{2}(O^2-2) > 0$ the hidden $\mu = 1$ parameter will preserve the sign and the type. From the natural condition $O \leq 2$ we have the boundary condition;

$$O_k = min\left(2, +\sqrt{\pm\frac{2+Jq}{qJ}}\right)$$

Given that $\pm\frac{2+Jq}{qJ} > 0$. Again, the q-opinion system foresees that under specific circumstances it happens that the interest related to the issue changes drastically.

4.3. The Multilevel Agreement

The q-XY model envisage also some general classes of the agreement. Despite the fact that the agreement vales O_x are continuous in its support, the average agreement can be grouped in distinct classes. It is interesting if we consider the commonly discrete behaviour in the society, say, individuals populating a given interval of the agreement, react discretely by admitting or refusing a proposal. To illustrate this theoretical envisagement of the model, let assume a 1D structure of random opinions interacting with a militant with the opinion O_1 aligned along the field F. each time fraction a pair would be created temporally. During the interaction, $O_{1,x} = 1$, Therefore, O_2 would tries (or is forced to) rotating itself to fulfil the equation;

$$(O_1 + O_2)_x = 2\langle O_x\rangle \tag{26}$$

Accordingly, by the amount of revolution needed to reach the matching configuration, the opinion values can be categorized in three groups related to the theoretical x-component $< \mathrm{O} >_{\mathrm{x}}$ which is calculated by the q-XY model and represent the thermodynamical or limit agreement with the question F issued on the system. The opinion with its $O_{2,x}$ component smaller than $2 * < \mathrm{O} >_{\mathrm{x}} - 1$ would be candidate to revolute by such a way that the x-component would have a random value in the interval $[Old.Value, 2 * < \mathrm{O} >_{\mathrm{x}} - 1]$. The second group having x-component in the interval $[2 * < \mathrm{O} >_{\mathrm{x}} - 1, < \mathrm{O} >_{\mathrm{x}}]$ would make a rotation to have $O_{2,x} = < \mathrm{O} >_{\mathrm{x}}$.The last group has nodes with $O_{2,x} > < \mathrm{O} >_{\mathrm{x}}$ so maybe they don't need to change their old position because they are already fitted in some extent with the F idea.

4.4. Stability Issue in Zero Degree

The specific opinions states corresponding to the critical points of (1) result from the solution of the simultaneous equations $\frac{\partial}{\partial O,\varphi}\left[-\frac{\mathrm{J}}{2}(\mathrm{O}^2-2)-\mathrm{FOcos\varphi}\left(1-\frac{\mathrm{qJ}}{2}(\mathrm{O}^2-2)\right)\right] = 0$. Note that the pair would arrive at the O_x agreement on the issue F by having established preliminarily an inner-opinion agreement O $\frac{2O_x}{cos\varphi} < O < 2$. Therefore, there are infinity states of the inner agreement that produce ascertain support on F. If both opinions are aligned, the $O = 2$, so $cos\varphi_1 = cos\varphi_2 = O_x$ the difference of the polar angles is $\Delta = \varphi_1 - \varphi_2 = 0$ but the same result could be found in the limit case $\varphi_2 = 0$, $\Delta = \varphi_1 - 0 = \varphi_1 = 1 - O_x$. It is clear now that various combination of the values (O, φ) are candidate to produce stationary states (if there would be). For the zero-temperature state we will refer the utility only. So, the q-XY utility has 9 critical points and therefore 9 temporal level of the stationary agreement are possible. In six of them, the total vector is aligned parallel with the field F, in one case it is orthogonal to it, and in two others the opinion vector has the directions given by the angle $2\mathrm{atan}\left(\left(\frac{2F^2q + 2F^2q^2 \pm 2.2^{\frac{1}{2}}F\left(q(q+1)\right)^{\frac{1}{2}} + 1}{2F^2q^2 + 2F^2q - 1}\right)^{\frac{1}{2}}\right)$. The natural requirement $O \leq 2$ imposes restrictions for q, F, J in stationary or near to-stationary states.

4.5. The No-interest State

Na important property emphasized by the q-model is related to the utility behaviour. Remember that ten utility function in our model is related to the common interest shared on the couple. The agreement level O and the rotating angle toward exterior field supposedly produced optimization of the utility or interest. Consider the thermodynamic utility given by Equation 16. For the thermodynamic states which concluded in the $O_x\,(J,q,F)$ and $U(J,q,F)$ calculated, we distinguish realizations O,φ where;

$$1-\frac{qJ}{2}(O^2-2)=0 \tag{27}$$

Recalling the Bessel function properties $I_1(0)=0$ and $I_0(0)=1$, one obtains $<O_x>=0$ and also $U=0$. It follows that the total opinion vectors having the magnitude;

$$O=\left(\frac{2+Jq}{Jq}\right)_+^{\frac{1}{2}}\leq 2 \tag{28}$$

are normal to the exterior field. Also, the utility of this state is null. The subscript (+) in (28) specify the cases where square root takes positive values. Let's assume now that from a random opinion configuration with zero component toward the field F, the opinions start the orientation by simply rotating toward x-axes. Throughout this rotation sequence during the physical thermalizing (cooling) process, if the condition (26) is meet, so;

$$O=\left(\frac{2+Jq}{Jq}\right)_+^{\frac{1}{2}}>2*<O_x> \tag{29}$$

The total opinion O will jump to the normal direction versus the exterior field. The q-XY model predicts *the switching to zero interest* or *suddenly ceased support* behaviour. In a critical moment, the two-individual x-components of those opinions become opponents toward the issue F producing a zero-level agreement for it! The solution of (28) restricts the inner conditions to follow $-1<q<-2$ for FM interaction and $1>q>2$ for AFM case. Therefore, the 'now *we don't care*" moment would be observable for certain couples which have their q-condition in a given interval as above. Note also that in this perspective the natural parameter of the system is the product q*J. If society is made up of heterogonous opinion pairs, the fraction of the community in such conditions would lose the interests on F for all q that enforces (28). We can assume that this phenomenon becomes observable for a certain zone around critical values. It is interesting that this state could not be shifted by simply changing exterior parameters F or T. In the same way, if we imagine a couple passing all conditions q, it would fall in the *no-interest* region for some specific parameter values. It looks that inner conditions affirm in some extend the subjectivity of the human behaviour that generate an ability for our system to evolve dramatically, producing for certain values the no-interests state.

5. The Modification of the Preferential Attachment Tendency

Let consider the q-opinion nodes interacting in a social network, for example in an electoral campaign. Theoretically, the probability of the link is given by the equation $p_{i.j}\sim k_j/sum(k_j)$ known as the preferential attachment rule [5-8], etc. The node (j) get known '*what about the node (j)"* and after being informed that it has k links, node (i) assume that all has been set rationally, based on the proper values of (j) and decides to attach it by the probability given above. However, in real live, there are more than those hardly and mechanical mechanisms involved. Firstly, the power law distribution predicted by the preferential attachment mechanisms is theoretical, in practice different distribution have been observed. Prenga and Ifti (2012) [23] proposed an ad hoc modification in the formula by including local field effects aiming in reproducing electoral distribution. The amended preferential attachment rule reads $p_{i.j}\sim h_j\left(k_j/sum(k_j)\right)$. Later we have proposed to employ the q-opinion findings to refine the above formula aiming to include rationality in decision making and the electoral supporting based on the precepted performance or utility. The idea is as follow: after contacting the candidate (j), the voter does realize his agreement or prior support for by 'calculating" the q-opinion given by Equation 15. Next, the voter contemplates the generalized rule of the political support given by the new formula;

$$p_{i.j}\sim<O_x>\ h_j\frac{k_j}{sum(k_j)} \tag{30}$$

To proceed with Equation 31 and Equation 15 on needs the interior q-parameter of the temporal pair voter-candidate. We proposed a specific form for this parameter to consider two following elements: the parameter q that energizes or discourages the agreement in the pair should depend on the electoral performance of the candidate. Also, we expect that if the performance of the candidate is very high, a non-zero probability should exist for non-natural support from voters of the opponent's wing affiliation. The electoral performance function is taken linear (vector product) $H=A*B$ where A are coefficients and B some socio-political performances. H can take values from zero to infinity in principle. The q-parameter value is proposed by the following formula;

$$q = J - \frac{2J}{1+exp(A*B)} \tag{31}$$

Remember that according to numerical assessment based on the Equation 15 the average thermodynamic opinion is higher for high value of $q * J$. So, if the electoral Hamiltonian (electoral performance) H is high and the voter contacted is a potential supporter ($J = 1$), we get $q = 1 - 2/(1 + inf) = 1$ and therefore the q-XY utility has the extra negative term which favorize a higher agreement level. If the electoral performance of the subject is too low, we get $q = 1 - 2/(1 + \exp(0)) = 0$ and the calculation reproduces the classical magnetic system. The interesting scenario is the case when the voter and the candidate are politically opponents ($J = -1$) and supposedly the agreement is high. We get $q = -1 + 2/(1 + inf) = -1$, so $q * J = 1$. The agreement level obtained by Equation 15 is not neglectable in those conditions. The model says that for very high performance of the candidates, there would be always a level of support: It is very high (practically 1) for normal temperature if the candidate belongs to the preferred parties, but it is always nonzero even for the candidates of the opponent side. If the performance is low ($H = A * B <<$), we obtain $q = -1 + 2/(1 + \exp(0)) = 0$, and the system produces a very low agreement level, practically zero for normal temperature (not too high). It can be clarified if we remember that like the antiferromagnetic case in the $T = 0$ limit, the opinions in the couple should be aligned antiparallel giving the total opinion zero, $\vec{O} = \vec{O}_1 - \vec{O}_2 = 0$. As a check of the model, we have reproduced the distribution of the votes obtained in the election of 2021 in Albania, where we have constated an increasing rational behavior of the voters and pragmatic voting since 2013. Various implantations are possible, but we highlight again that q-opinion models reveal realistic electoral behaviors that are difficult to describe in the framework of standard opinion models.

6. Hamiltonian Dynamics Approach

In the framework of mechanical dynamics, we can analyze the time evolution of the opinion in the q-pair. In this case we propose to include a virtual kinetic term $U_k = \sum_{i=1}^{2} I \frac{\dot{\varphi}^2}{2}$ in the Hamiltonian (3) similarly with discussion in Leoncini et al. (1998) [24]. Next, from the standard Hamiltonian dynamics, we have formally $\dot{L_\iota} = \frac{\partial U}{\partial \varphi_i}$ and $\dot{\varphi_\iota} = \frac{\partial U}{\partial p_i}$. Without losing generality we take m=1, so $p_\varphi = mO^2\varphi^{\cdot} \equiv O^2\dot{\varphi}$.

The extended utility now reads;

$$U_p = -\frac{J(O^2-2)}{2} - FOcos\varphi + q\frac{J(O^2-2)}{2} * FOcos\varphi + I\frac{\dot{\varphi}^2}{2} + \frac{m\dot{O}}{2} \tag{32}$$

And now the variables are (O, φ) form Hamiltonian equations we have;

$$\dot{p_\iota} = \frac{\partial U}{\partial \varphi_i} = FOsin\varphi - q\frac{J(O^2-2)}{2} * FOsin\varphi \tag{33}$$

$$(I\varphi')' = \frac{\partial U}{\partial \varphi_i} = FOsin\varphi - q\frac{J(O^2-2)}{2} * FOsin\varphi$$

In this case we have the following equation for the total opinion and its component:

$$O^2\ddot{\varphi} + 2O\dot{O}\dot{\varphi} = FOsin\varphi - q\left(\frac{J}{2}(O^2-2) * FOsin(\varphi)\right)$$

So;

$$O\ddot{\varphi} = Fsin\varphi - q\left(\frac{J}{2}(O^2-2) * Fsin(\varphi)\right) - 2\dot{O}\dot{\varphi}$$

And the other equation reads;

$$\ddot{O} = -JO - Fcos\varphi + FJqcos\varphi\left(\frac{3O^2}{2} - 1\right)$$

To conclude with the numerical integration of those equations and to display the time dynamics of the magnitude of the total opinion vector O [25], we rewrite them as linear systems;

$[[\dot{\varphi} = \omega]]$

$$\dot{\omega} = \frac{\left(Fsin\varphi - q\left(\frac{J}{2}(O^2-2) * Fsin(\varphi)\right) - 2\dot{O}\omega\right)}{O}$$

$$O' = \emptyset$$

$$\dot{\emptyset} = -JO - Fcos\varphi + FJqcos\varphi\left(\frac{3O^2}{2} - 1\right) \tag{34}$$

which can be easily integrated numerically. So, we have observed that the dynamics of the opinion is usually oscillatory and the opinion changes its values with time, whereas agreement on the exterior issue tends to its thermodynamic value given by Equation 12.

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The author declares that there is no conflict of interests regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the author.

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